PUPT-1674

Inflationary Textures

Neil Turok[†,‡] **and Yong Zhu**[†]

[†]Physics Department

Princeton University, Princeton NJ 08544,

[‡]DAMTP

Silver Street, Cambridge, U.K.

Abstract

We present a calculation of the power spectrum generated in a classically symmetry-breaking $O(N)$ scalar field through inflationary quantum fluctuations, using the large-N limit. The effective potential of the theory in de Sitter space is obtained from a gap equation which is exact at large N. Quantum fluctuations restore the $O(N)$ symmetry in de Sitter space, but for the finite values of N of interest, there is symmetry breaking and phase ordering after inflation, described by the classical nonlinear sigma model. The main difference with the usual cosmic texture scenario is that the initial conditions possess long range correlations, with calculable exponents.

12/96

Email address: [‡]N.G.Turok@damtp.cam.ac.uk, [†]zhu@puhep1.princeton.edu.

1. Introduction

There are currently two broad categories of theories of cosmological perturbation: those which generate structure via quantum fluctuations during inflation [1], and those which generate structure upon the process of symmetry breaking and field ordering through the Kibble mechanism [2]. Inflation is a brief phase of superluminal expansion in the early universe, which is invoked to solve the horizon problem, the flatness problem and other puzzles in the standard cosmology. The existence of an event horizon gives rise to quantum fluctuations in the scalar field, which can be attributed to the Hawking temperature of de Sitter space. These fluctuations are then stretched out to cosmological scales. As the wavelength of a given mode expands beyond the Hubble radius H^{-1}, its amplitude becomes frozen. A broad-band spectrum of fluctuations with this amplitude is generated over the course of the inflation. Once inflation is over, the Hubble radius grows faster than the scale factor $a(t)$. The fluctuations stretched out of the the Hubble radius during inflation re-enter after inflation with nearly same amplitude, giving an almost scale-invariant spectrum of perturbations [3],[4]. Requiring that the density perturbations are consistent with the observed level of structure in today's universe requires some fine-tuning of parameters in most inflationary models.

The symmetry breaking category [2], includes the theories of cosmic strings[5], global monopoles and textures [6]. A simple class of theories with broken global symmetry is provided by the "$O(N)$" models, where a global $O(N)$ symmetry group is spontaneously broken to $O(N-1)$ by a N-component scalar field $\vec{\phi}$. For $N = 1$, one has domain walls, $N = 2$, global strings and $N = 3$, global monopoles. For $N = 4$, there are textures, and for $N > 4$, "nontopological textures". In the limit of large N, there are no topological defects, and the theory is exactly solvable [7]. In these pictures, the universe is usually assumed to begin in a hot, smooth and homogeneous state, perhaps created by inflation or just special initial conditions. As the universe cools, the symmetry is broken and a disordered phase forms, containing defects of the appropriate type. As the universe expands, the tangle of defects is ordered on progressively larger scales. Density perturbations of constant amplitude at horizon crossing are generated in the process, resulting in a scale invariant power spectrum of perturbations. The phase ordering process involves nonlinear physics,

which leads to a nongaussian pattern of fluctuation [8], [9] through which they may be distinguished from inflation.

At the current stage of a rapidly developing field, none of the simplest versions of the above theories provides a compelling match to the observations. Several variations of the simplest inflationary model have been introduced in an attempt to improve matters: tilting the power spectrum, introducing gravity waves, adding hot dark matter or a cosmological constant, or considering open inflation. It is also surely worthwhile to consider other options. One of the simplest is just to combine the topological defect and inflation pictures. One possibility is to have standard inflation, with defects produced after reheating in the usual manner. Such a theory would have inflationary and defect-induced fluctuations, in some linear combination, uncorrelated with each other. A detailed study of reheating after inflation is recently been introduced by Kofman, Linde, and Starobinsky [10], where parametric resonance may rapidly transfer most of the energy of an inflaton field to the energy of other bosons. This process may result in copious production of topological defects. Perhaps more interesting is the case where exponentially large defects are generated through phase transitions during the last stages of inflation [4]. Vishniac, Olive and Seckel [11] have proposed a class of models in which the inflaton is coupled to the string-producing field. And the strings are formed late in inflation as the inflaton rolls towards its zero-temperature value. The case that cosmic strings and textures are produced in inflation via quantum creation or sufficiently high reheat temperature has been considered by Hodges and Primack [12]. A second-order phase transition during inflation is studied as a natural mechanism to produce topological defects by Nagasawa snd Yokoyama [13]. Finally, quantum creation of topological defects through instanton tunneling during inflation has been investigated by Basu, Guth, and Vilenkin [14].

In this work, we aim at an analytic treatment of this problem using large N techniques, developing earlier work by R. Davis and one of us [15]. We study a renormalisable symmetry beaking $O(N)$ scalar field theory in the large N limit. We study the phenomenon of symmetry restoration in de Sitter space [16], and match the field modes to those in the subsequent radiation dominated era. This provides a set of initial conditions for the usual sigma model evolution, but with long range correlations. The primordial scalar field power

spectrum P_k is obtained as a function of the scalar self-coupling.

2. $\lambda\phi^4$ Theory in de Sitter Space

We study the $\lambda\phi^4$ theory in four dimensions with the lagrangian density

$$L = \sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi^\alpha\partial_\nu\phi^\alpha - \tfrac{1}{2}(m_0^2 + \xi_0 R^2)|\vec{\phi}|^2 - \frac{\lambda_0}{4!N}|\vec{\phi}|^4\right] . \tag{1}$$

2.1 Large N Approximation

The $1/N$ expansion of the effective potential is constructed by summing the "super-daisy diagrams" [17] in the limit of $N \to \infty$ with fixed coupling constant λ. These formulas typically display richer structures than the corresponding leading-order expressions in ordinary perturbation theory because that the leading $1/N$ approximation preserves much more of the nonlinear structure of the exact theory than does ordinary lowest-order perturbation theory. Large N behavior of a massless scalar field in flat spacetime was first studied by Coleman, Jackiw and Politzer [18]. As well known, the one-loop radiative correction to the effective theory dynamically breaks the $O(N)$ symmetry [19]. However, a tachyonic excitation was found in the symmetry breaking state at large N, which cast some doubts on the validity of the large N approximation. Later, Abbott, Kang and Schnitzer [20] and Linde [21] pointed out that the effective potential of the scalar field ϕ is a double-valued function, with one branch containing the symmetry breaking phase and the other including the symmetric solution. It was shown that the symmetric state *always* had lower effective energy. Thus the ground state of the theory is $O(N)$-symmetric, and the spontaneous symmetry breaking is not possible in the large-N limit. No tachyon exists in the symmetric vacuum, and the excited states can be studied through perturbative methods.

This poses a problem for us - in the large N limit, the theory we are discussing does not have a symmetry breaking phase, even in Minkowski space. So there would be no classical phase ordering following inflation. However, we shall take the attitude that we are not in any case really interested in very large N, (not least because the amplitude of density perturbations goes to zero as $1/\sqrt{N}$), but we merely wish to use the large N approximation as a tool to approximate a theory with $N = 3, 4$ or 5. We know that at these modest values of N the approximation *fails* in Minkowski space, so there is symmetry

breaking. It is nevertheless reasonable to hope that the approximation *is* valid in de Sitter space, where one expects symmetry restoration to occur for any $N > 1$ [16]. We shall therefore treat the problem in the quantum mechanical large N approximation in de Sitter spaces, but the classical large N approximation (which does have symmetry breaking) in the subsequent radiation era. This may seem a schizophrenic point of view, but it is one we are forced to in order to have a chance of reproducing even the qualitative physics correctly.

2.2 de Sitter Ground State

We work in the flat slicing of de Sitter spacetime with conformal coordinate η and metric

$$d\tau^2 = a^2(\eta)[d\eta^2 - (d\mathbf{x})^2] \ . \tag{2}$$

where the scale factor takes $a(t) = e^{Ht}$, or

$$a(\eta) = -\frac{1}{H\eta}, \quad \eta < 0 \ . \tag{3}$$

We rescale $a(\eta)$ to be unity in the beginning of inflationary epoch. Thus at the end of inflation, or the transition to the radiation dominated era, we have $\eta_0 = -\eta_{transition} \ll 1/H$.

In curved spacetime, the massive scalar field with mass m can be expanded as

$$\phi^\alpha(\mathbf{x},\eta) = (2\pi)^{-3/2} \int d^3k[\phi_k(\eta)\hat{a}^\alpha_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \phi^*_k(\eta)\hat{a}^{\alpha\dagger}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}] \ , \tag{4}$$

where

$$[\hat{a}^a_{\mathbf{k}}, \hat{a}^{b\dagger}_{\mathbf{q}}] = (2\pi)^3 \delta^{ab} \delta^3(\mathbf{k} - \mathbf{q}) \ . \tag{5}$$

The mode function $\phi_{_k}(\eta)$ can be written in a separated form as [22] $\phi_{_k}(\eta) = a^{(2-n)/2}\chi_{_k}(\eta)$. The canonical quantization conditions reduce to a condition on the Wronskian of the solutions $\chi_{_k}$: $\chi_{_k}\partial_\eta\chi^*_{_k} - \chi^*_{_k}\partial_\eta\chi_{_k} = i$. As usual, we user the Bunch-Davies vacuum [23] for the scalar field:

$$\begin{aligned} \phi_k(\eta) &= \sqrt{\frac{\pi}{4}} H\eta^{3/2} H^{(2)}_\nu(k\eta) \ , \\ \nu^2 &= \frac{(n-1)^2}{4} - \frac{m^2}{H^2} \ , \end{aligned} \tag{6}$$

where n is the spacetime dimension. The two-point correlation function can be expressed in dimensional regularization [24]

$$G(x,x) = \frac{H^2}{8\pi^2}\frac{\Gamma(\nu(n)-\frac{1}{2}+n/2)\Gamma(-\nu(n)-\frac{1}{2}+n/2)}{\Gamma(\frac{1}{2}+\nu(n))\Gamma(\frac{1}{2}-\nu(n))}\Gamma(1-n/2)\ . \tag{7}$$

2.3 $1/N$ Approximation in de Sitter Space

The summation of superdaisy diagrams in flat space time has been discussed extensively [25]. The similar calculation is carried out here in de Sitter metric. We merely state the gap equation here, which follows from identical diagrammatics to the flat spacetime case

$$2\frac{dV_{eff}(\phi^2)}{d\phi^2} \equiv M_{ds}^2(\phi^2) = m_0^2 + \xi_0 R + \frac{\lambda_0}{6N}\phi^2 + \frac{\lambda_0}{6}\langle\phi^2\rangle|_{M_{ds}}\ , \tag{8}$$

where M_{ds}^2 is the effective mass squared of the theory in de Sitter space. Note that in the gap equation (8) ϕ is the classical field, while $\langle\phi^2\rangle$ is the two-point function given by equation (7). From equation (7), the two-point function diverges at $n=4$. Its renormalization is carried out in the standard procedure through dimensional regularization and minimal subtraction [24]:

$$\begin{aligned}
m^2 &= m_0^2 + \frac{\lambda_0}{6}\frac{|m|^{d-2}}{(4\pi)^{d/2}}\Gamma(1-d/2)\ ,\\
\frac{1}{\lambda} &= \frac{1}{\lambda_0} + \frac{1}{6}\frac{|m|^{d-4}}{(4\pi)^{d/2}}\Gamma(2-d/2)\ ,\\
\xi &= \xi_0 + \frac{\lambda_0}{6}d(d-1)(\frac{1}{6}-\xi)\frac{|m|^{d-4}}{(4\pi)^{d/2}}\Gamma(2-d/2)\ .
\end{aligned} \tag{9}$$

We shall only study the simplest case: zero conformal coupling, since the extension to nonzero ξ is straightforward.

An alternative approach to remove the divergence in the two-point function is to apply an effective cutoff on the comoving momentum due to the inflationary expansion [26]. We shall take this approach to verify the consistency and validity of renormalization scheme in our model.

After renormalisation, the gap equation (8) reads

$$M_{ds}^2 = m^2 + \frac{16\pi^2}{N} g\phi^2 + \frac{1}{2} g M_{ds}^2 ln \frac{M_{ds}^2}{|m^2|} - 12 g H^2 ln \frac{M_{ds}^2}{|m^2|} + g H^2 \Bigg\{ (\frac{M_{ds}^2}{H^2} - 2) \left[\Psi(\frac{3}{2} + \nu) + \Psi(\frac{3}{2} - \nu) - ln \frac{M_{ds}^2}{H^2} - 1 \right] + \frac{M_{ds}^2}{H^2} - \frac{2}{3} \Bigg\} , \tag{10}$$

where

$$g = \frac{\lambda}{96\pi^2 + \lambda}, \quad \nu^2 = \frac{9}{4} - \frac{M_{ds}^2}{H^2} \ . \tag{11}$$

The effective potential can be computed by integrating M_{ds}^2

$$\begin{aligned} V(\Phi) &= \int_0^\Phi \frac{dV}{d\phi} d\phi \ + \ V(0) \\ &= \int_0^\Phi \phi \cdot M_{ds}^2(\phi^2) \cdot d\phi \ + \ V(0). \end{aligned} \tag{12}$$

3. FRW Solution and Mode Matching

In the radiation dominated era, the field ordering is described classically by the non-linear sigma model [7] . The evolution of the N-component scalar field $\vec{\phi}$ obeys

$$\partial_\eta^2 \phi^\alpha + 2 \frac{\partial_\eta a(\eta)}{a(\eta)} \partial_\eta \phi^\alpha - \vec{\nabla}^2 \phi^\alpha = - \frac{(\partial \vec{\phi})^2}{\phi_0^2} \phi^\alpha \equiv \frac{T_0}{\eta^2} \phi^\alpha. \tag{13}$$

In the large N approximation, T_0 may be treated as a constant once scaling behaviour is attained for the field ordering. In momentum space, the decoupled mode functions satisfy

$$\frac{d^2\phi_k}{d(k\eta)^2} + \frac{\alpha}{k\eta} \frac{d\phi_k}{d(k\eta)} + (1 - \frac{T_0}{(k\eta)^2}) \phi_k = 0, \tag{14}$$

where $\alpha = 2 \frac{dln(a(\eta))}{dln\eta}$ with $\alpha = 2$ for radiation-dominated era and $\alpha = 4$ for matter-dominated era. Note that equation (14) is symmetric under $\eta \to -\eta$. In the radiation dominated era, the solution of (14) can be generally expressed as

$$\begin{aligned} \phi_k &= \sqrt{\frac{\pi}{4}} H(-\eta)^{-\frac{1}{2}} \left[C_1(k) H_\mu^{(1)}(-k\eta) + C_2(k) H_\mu^{(2)}(-k\eta) \right], \\ \mu^2 &= T_0 + \frac{1}{4} \ . \end{aligned} \tag{15}$$

After the end of inflation, the mode functions (6) of ϕ in de Sitter space is matched to the FRW classical solution (15) by requiring that scale factor $a(\eta)$, $\phi_k(\eta)$, and $\dot{\phi}_k(\eta)$ be continuous at the transition point $\eta = -\eta_0$. A simple model of the transition can be constructed by considering a metric of the form

$$a(\eta) \;=\; \begin{cases} -(H\eta)^{-1}, & \eta < -\eta_0, \quad \text{de Sitter} \\ (\eta + 2\eta_0)/H\eta_0^2, & \eta > -\eta_0, \quad \text{FRW} \end{cases} \tag{16}$$

and the matching condition reads

$$\begin{aligned} {\eta_0}^2 H_\nu^{(2)}(k\eta_0) &= \Big[C_1 H_\mu^{(1)}(k\eta_0) + C_2 H_\mu^{(2)}(k\eta_0) \Big] \,, \\ \eta_0^2 \dot{H}_\nu^{(2)}(k\eta_0) + \frac{3}{2}\eta_0 H_\nu^{(2)}(k\eta_0) &= \Big[C_1 \dot{H}_\mu^{(1)}(k\eta_0) + C_2 \dot{H}_\mu^{(2)}(k\eta_0) \Big] \\ &\quad - \frac{1}{2\;\eta_0} \Big[C_1 H_\mu^{(1)}(k\eta_0) + C_2 H_\mu^{(2)}(k\eta_0) \Big] \,, \end{aligned} \tag{17}$$

where dot denotes $\partial/\partial\eta_0$. The solution of (17) is expressed as

$$\begin{aligned} C_1 &= -\frac{i\pi\,\eta_0}{4} \Big[\eta_0^2 H_\nu^{(2)}(k\eta_0) B - H_\mu^{(2)}(k\eta_0) C \Big] \,, \\ C_2 &= \frac{i\pi\,\eta_0}{4} \Big[\eta_0^2 H_\nu^{(2)}(k\eta_0) A - H_\mu^{(1)}(k\eta_0) C \Big] \,, \end{aligned} \tag{18}$$

where the coefficients read

$$\begin{aligned} A &= \dot{H}_\mu^{(1)}(k\eta_0) - \frac{1}{2\eta_0} H_\mu^{(1)}(k\eta_0) \,, \\ B &= \dot{H}_\mu^{(2)}(k\eta_0) - \frac{1}{2\eta_0} H_\mu^{(2)}(k\eta_0) \,, \\ C &= \eta_0^2 \dot{H}_\nu^{(2)}(k\eta_0) + \frac{3}{2}\eta_0 H_\nu^{(2)}(k\eta_0) \,. \end{aligned} \tag{19}$$

Easily we can verify the normalization condition for Bunch-Davis vacuum. The normalization conditions for the FRW mode functions are also automatically satisfied.

The mode function (15) in FRW can be separated into a 'classical' part (by which we mean the component which survives with large amplitude long after the transition to radiation domination) and a 'quantum' part (which diminishes quickly after inflation):

$$\begin{aligned} \phi_k(\eta) \;&\equiv\; \phi_{cl} \;+\; \phi_q \;, \\ &=\; a(k)\; J_\mu(k\eta) \;+\; b(k)\; Y_\mu(k\eta) \end{aligned} \tag{20}$$

,

where the coefficients a and b write

$$\begin{aligned} a(k) \;&=\; H\frac{\pi^{\frac{3}{2}}\,\eta_0}{8}\left[\eta_0^2\,H_\nu^{(2)}(k\eta_0)\;(2\dot{Y}_\mu(k\eta_0)-\frac{Y_\mu(k\eta_0)}{\eta_0})\;-\;2\,C\,Y_\mu(k\eta_0)\right]\,, \\ b(k) \;&=\; -H\frac{\pi^{\frac{3}{2}}\,\eta_0}{8}\left[\eta_0^2\,H_\nu^{(2)}(k\eta_0)\;(2\dot{J}_\mu(k\eta_0)-\frac{J_\mu(k\eta_0)}{\eta_0})\;-\;2\,C\,J_\mu(k\eta_0)\right]\,. \end{aligned} \tag{21}$$

At the epoch of most importance for structure formation, or when a given mode re-enters the Hubble radius, $k\eta \approx 1$, the 'classical' part dominates the 'quantum' part by a factor of $\left|\frac{\eta}{\eta_0}\right|^{2\mu}$ with $\mu > 0$. Consider how small η_0 is, this is an extremely large number. It is therefore reasonable to neglect the quantum part of the wavefunction at FRW space. Therefore, the classical non-linear sigma model is a valid approximation to the $O(N)$ scalar field at FRW epoch.

After matching the quantum fluctuations of the scalar field to the classical evolution of the nonlinear sigma model, we can now relate the original parameters in the quantum field theory to the value of $\vec{\phi}^2 = \phi_0^2$ in the classical theory. We have

$$\mu = 2-\nu\ . \tag{22}$$

And at large $\eta \gg \eta_0$

$$\begin{aligned} \phi_0^2 \;\equiv\; N\langle\phi_{cl}^2\rangle \;&=\; N\int\frac{d^3k}{(2\pi)^3}\frac{1}{\eta}\phi_{cl}^2\ , \\ &= N\frac{H^2}{4\pi^4}\frac{\Gamma^2(\nu)\Gamma^2(\mu)\Gamma(\mu-1/2)}{\Gamma(\mu+3/2)}\ . \end{aligned} \tag{23}$$

At later time in FRW, the classical dynamics of the $\lambda\phi^4$ theory is totally fixed by the tree diagrams. Thus, the renormalised values m^2 and λ are fixed by

$$m^2 \;=\; -\frac{\lambda}{6N}\phi_0^2(H)\ , \tag{24}$$

where $\phi_0^2(H)$ is a function of H as fixed by (23). To re-emphasize the point, we are treating the theory as classical in the post-inflation era, with the given renormalised parameters, with the justification that for modest values of N and λ this should be a reasonable approximation.

4. Effective Potential and Vacuum in de Sitter Spacetime

The effective potential of the $\lambda\phi^4$ theory in de Sitter space is studied under the assumption of equation (24). After plugging (24) into the gap equation (10), our theory will have only one free parameter — the renormalised coupling constant λ. The effective potential V_{eff} is obtained for different regions of λ and the corresponding vacuum states are found by minimizing V_{eff}

$$\begin{aligned} 0 = \frac{dV_{eff}(\phi^2)}{d\phi} &\equiv 2\phi \frac{dV_{eff}}{d\phi^2} \\ &= \phi\, M_{ds}^2(\phi^2)\ . \end{aligned} \tag{25}$$

To satisfy equation (25), the vacuum state will either have $|\phi| = 0$, where the symmetry is restored or $M_{ds}^2 = 0$, where the symmetry is broken.

4.1 Weak coupling: $\lambda \ll N$

Weak coupling implies the small mass limit: $|M_{ds}^2| \ll H^2$. At first, let's study the weak interacting limit of the gap equation (10) with $\lambda \ll N$ and $|M_{ds}^2| \ll H^2$. The leading order terms in this limit

$$\begin{aligned} \nu &= \sqrt{\frac{9}{4} - \frac{M_{ds}^2}{H^2}} = \frac{3}{2} - \frac{1}{3}\frac{M_{ds}^2}{H^2} + \mathrm{O}(\frac{M_{ds}^2}{H^2})\ , \\ \phi_0^2 &= \frac{3NH^2}{16\pi^2}\frac{H^2}{M_{ds}^2}\ , \\ m^2 &= -\frac{\lambda}{6N}\phi_0^2 = -\frac{\lambda H^2}{32\pi^2}\cdot\frac{H^2}{M_{ds}^2}\ , \\ \psi(\frac{3}{2} - \nu) &= \psi(\frac{M_{ds}^2}{3H^2}) = -\frac{3}{2}\frac{H^2}{M_{ds}^2}\ . \end{aligned} \tag{26}$$

The leading order of the gap equation (10) is

$$\frac{M_{ds}^2}{H^2} = \sqrt{\frac{\lambda}{32\pi^2}}\ . \tag{27}$$

The variable $\nu(\lambda)$ can be expressed as

$$\nu = \frac{3}{2} - \frac{1}{12\pi}\sqrt{\frac{\lambda}{2}}\ .$$

Generally, the relation of $M_{ds}^2(\phi^2)$ over ϕ^2 is plotted at Fig.(1). At the region $M_{ds}^2 < A$, ϕ^2 is negative, which is obviously unphysical. Thus, M_{ds}^2 cannot be zero and the ground state of the scalar field is the $O(N)$-symmetric state

$$|\phi| = 0, \quad M_{ds}^2 = A > 0\ . \tag{28}$$

The effective potential is obtained by integrating M^2_{ds}, as shown qualitatively in Fig.(2). For small ϕ, V_{eff} can be easily calculated

$$\begin{aligned} V_{eff}(\phi) \; &= \int_0^{\phi} \phi \; M^2_{ds} \; d\phi \sim \int_0^{\phi} \phi(A \; + \; \frac{16\pi^2 g}{N}\phi^2) \; d\phi \; , \\ &= \frac{A}{2}\phi^2 \; + \; \frac{4\pi^2}{96\pi^2 \; + \; \lambda} \frac{\lambda}{N}\phi^4 \; . \end{aligned} \tag{29}$$

For $\phi > \phi_{max}$, the effective mass squared M^2_{ds} turns complex and the effective potential has an imaginary part, which is related to the particle creation in the scalar field ϕ [21]. The imaginary V_{eff} and the vacuum instability imply that one can actually deal with the vacuum state with the field ϕ only during such a short time interval, for which the energy density of the particles created during this time, will be much less than Re(V_{eff}), and the effects of the tachyonic instability of vacuum can be neglected.

The primordial power spectrum of the scalar field is obtained by taking the long-wavelength limit of the two point correlation function,

$$\begin{aligned} P(k,\eta) &= \; \langle \phi^{\alpha}_k(\eta)\phi^{\alpha}_{-k}(\eta)\rangle \; , \\ &= \; a(k)^2 J^2_{\mu}(k\eta) \; , \\ &\sim \; \frac{1}{k^{2\nu}} \; , \\ &= \; k^{-3+\frac{2}{3}\frac{M^2_{ds}}{H^2}} \xrightarrow{\lambda \to 0} \; k^{-3+\frac{1}{6\pi}\sqrt{\frac{\lambda}{2}}} \; . \end{aligned} \tag{30}$$

4.2 Intermediate Coupling: $\lambda \sim N$

As λ is increased to the order of of N, the effective mass squared M^2_{ds} turns to be a double-valued function of ϕ^2, as illustrated in Fig.(3). A new $O(N)$-symmetric solution C emerges with $M^2_{ds} \sim H^2$. Vacuum solution C has a lower effective potential compared to vacuum A. This can be shown by taking the integration of M^2_{ds} from A to the transition point B and then from B to C

$$V(C) \; = \int_A^B \phi M^2_{ds}(\phi) d\phi \; + \int_B^C \phi M^2_{ds}(\phi) d\phi + V(A) \; < \; V(A) \; . \tag{31}$$

Thus, state C is the true ground state, as illustrated in Fig.(4). The large effective mass vacuum state is an unique solution in the large N approximation, which is absent in the

traditional one-loop analysis [27]. At the limit of $\frac{M_{ds}^2}{H^2} \to \frac{9}{4}$, we obtain almost Gaussian white noise spectrum with $\nu \to 0$

$$\frac{M_{ds}^2}{H^2} = \frac{9}{4} - \nu^2, \qquad m^2 = -\frac{\lambda H^2}{90\pi^4}\frac{1}{\nu^2} \ .$$

The leading order of the gap equation (10) is

$$\nu^2 \log(\nu) + \frac{4(96\pi^2 + \lambda)}{87 \cdot 90\pi^4} = 0 \ . \tag{32}$$

Numerical simulations show that small ν solution is obtained around $\lambda \sim N$ and the vacuum state C exists at the range of $0 < \nu < 0.8$. And the power spectrum is

$$P_k \approx \frac{1}{k^{2\nu}} \approx \frac{1}{k^{0\,\sim\,1.6}} \ .$$

For very large λ, or $\lambda \gg N$, the contribution from high order terms will be significant and the the $1/N$ expansion is no longer reliable [21]. The power spectrum index 2ν versus λ is plotted in Fig.(5).

5. Verification of the Renormalization and Matching Schemes

The divergence of two-point function $\langle\phi^2\rangle$ in de Sitter space can be removed by a less formal but more intuitive methods: inflationary stage naturally provides both ultraviolet and infrared cutoffs at momentum space. Let's consider the physical wavelength of mode k, $\lambda_{phy} \equiv a(\eta)/k$. Any mode with physical wavelength greater than the Hubble radius at the beginning of inflation ($\eta = \eta_i$), or

$$\lambda_{phys} > 1/H \ \Rightarrow \ k < H, \quad \text{with } \eta_i = -\frac{1}{H} \ \text{ and } \ a(\eta_i) = 1 \ , \tag{33}$$

would be totally stretched out by inflation and is irrelevant to the physical processes after inflation. Also, any mode function with physical wavelength shorter than the Hubble radius after inflation will get more and more subhorizon in FRW, so it does not contribute to $\langle\phi^2\rangle$ either,

$$\lambda_{phys} < 1/H \ \Rightarrow \ k > 1/\eta_0 \quad . \tag{34}$$

The same scheme was applied by Vilenkin [26] to study the quantum fluctuation of a scalar field during inflation.

We are most interested in the small coupling region, where $\frac{M_{ds}^2}{H^2} \ll 1$ and $\nu \sim \frac{3}{2} - \frac{1}{3}\frac{M_{ds}^2}{H^2}$. From equation (6)

$$\begin{aligned}
\langle\phi^2\rangle = N \int_H^{1/\eta_0} \frac{d^3k}{(2\pi)^3}\phi_k^2(k\eta_0) &= \frac{N\pi H^2}{4}\int_H^{1/\eta_0} \frac{d^3k}{(2\pi)^3}\eta_0^3\Big[H_\nu^{(2)}(k\eta_0)\Big]^2 \ , \\
&\sim \frac{N}{16}\frac{H^2\Gamma^2(\nu)4^{2\nu}}{\pi^2}\int_0^1 d(k\eta_0)(k\eta_0)^{2-2\nu} \ , \\
&\sim 4N\frac{H^2\Gamma^2(\nu)}{\pi^2}\frac{1}{3-2\nu} \ .
\end{aligned} \tag{35}$$

In FRW space, the mode functions satisfy equation (23)

$$\begin{aligned}
\phi_0^2 &\sim N\frac{2H^2\nu^2}{\pi^3}(\Gamma(\nu)\Gamma(\mu))^2\frac{\Gamma(\mu-1/2)}{\Gamma(\mu+3/2)} \ , \\
&\sim \frac{9N}{2}\frac{H^2\Gamma^2(\nu)}{\pi^2}\frac{1}{3-2\nu} \ .
\end{aligned} \tag{36}$$

The $\langle\phi^2\rangle$ terms match remarkably well from de Sitter stage to FRW stage in this limit.

Around the Gaussian white noise limit, we have $\frac{M_{ds}^2}{H^2} \sim 9/4, \quad \nu \sim 0, \quad$ and $\phi_\nu(x) \sim \frac{i\pi}{2}ln(\frac{2}{x})$. The de Sitter solution has

$$\langle\phi^2\rangle = N\int_H^{1/\eta_0}\frac{d^3k}{(4\pi)^3}\phi_k^2(k\eta_0) \sim \frac{NH^2}{16\pi^2}\int_0^1 d(k\eta_0)(k\eta_0)^2 ln^2(\frac{2}{k\eta_0}) \ . \tag{37}$$

And the mode functions of FRW satisfy equation (23)

$$\begin{aligned}
\phi_0^2 &\sim N\frac{H^2}{\pi^2}\int\frac{d(k\eta)}{(k\eta)^2}ln^2(\frac{2}{\mathrm{k}\eta_0})J_\mu^2(k\eta_0) \ , \\
&\sim \frac{NH^2}{64\pi^2}\int d(k\eta_0)(k\eta_0)^2 ln^2(\frac{2}{k\eta_0}) \ .
\end{aligned} \tag{38}$$

Equation (37) and equation (38) are dominated by small argument integrand, so they match each other relatively well.

The consistency among above two schemes of renormalisation verifies the soundness of our approaches.

6. Conclusion

In conclusion of the paper, topological defect originated from quantum fluctuations during inflation is a viable and promising picture for cosmic structure formation. We show that the de Sitter vacuum state provides a natural initial condition for the symmetry breaking scalar field in FRW, which is described by the classical nonlinear sigma model. Due to an early stage of inflation, the scalar field possesses superhorizon correlations. Thus, this scenario might have more large scale power than usual cosmic defects models. By proper adjustment of the only free parameter: the coupling constant λ, we can obtain power spectrum ($P_k = k^{-2\nu}$) ranging from exact scale-invariant to Gaussian white noise. In the weak coupling case ($\lambda \ll N$), we have $2\nu = 3-\delta$, where $\delta \sim \frac{1}{6\pi}\sqrt{\frac{\lambda}{2}}$. For intermediate coupling region ($\lambda \sim N$), 2ν takes the value of 0 to 1.6.

7. Acknowledge

We would like to thank M. Bucher, A.D. Linde for helpful comments and discussion. This work was supported by NSF contract PHY90-21984, and the David and Lucile Packard Foundation.

Figure Captions

Figure 1: The effective mass squared M_{ds}^2 versus ϕ in the limit of weak coupling with $\lambda \ll N$. At the symmetric phase point A: $\phi^2 = 0$ and $M_{ds}^2(A) > 0$.

Figure 2: The effective potential V_{eff} in the weak-coupling limit. The effective potential is calculated by integrating M_{ds}^2. The $O(N)$-symmetric ground state lies in $\phi^2 = 0$. For $\phi^2 > \phi_{max}^2$, the effective potential turns imaginary representing the generation of classical field.

Figure 3: The effective mass M_{ds}^2 versus ϕ in the intermediate interaction range with $\lambda \sim N$. New vacuum solution C emerges in the limit of large effective mass. The positive $|\phi|^2$ region is divided into two separated regions, from A to B, and from B to C.

Figure 4: The effective potential V_{eff} in the intermediate coupling region. The $O(N)$-symmetric ground state is the large mass solution C which has lower effective potential.

Figure 5: Power spectrum index 2ν with respect to λ. ($P_k = k^{-2\nu}$) At $\lambda \ll N$, we have the almost scale-invariant spectrum $2\nu \sim 3$. The vacuum state jumps from A to C when λ reachs the order of N. Gaussian white noise limit is obtained around $\lambda \sim N$. There is no solution for $\lambda \gg N$.

References

[1] A.A. Starobinsky, Phys. Lett. **91B**, 99 (1980); A.H. Guth, Phys. Rev. **D23**, 347 (1981); A.D. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[2] T.W.B. Kibble, J. Phys. **A9**, 1387 (1976).

[3] S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A.A. Starobinsky, Phys. Lett. **117B**, 175 (1982); A.H. Guth and S.Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); J. Bardeen, P. Steinhardt and M. Turner, Phys. Rev. **D28**, 679 (1983).

[4] L.A. Kofman and A.D. Linde, Nucl. Phys. **B282**, 555 (1986).

[5] For a review see A. Vilenkin and E.P.S. Shellard, "*Cosmic Strings and other Topological Defects*", Cambridge University Press, 1994.

[6] N. Turok, Phys. Rev. Lett. **63**, 2625 (1989).

[7] N. Turok and D. Spergel, Phys. Rev. Lett. **66**, 3093 (1991).

[8] D. Coulson, P. Ferreira, P. Graham, and N. Turok, Nature **368**, 27 (1994).

[9] N. Turok, DAMTP preprint, bulletin board astro-ph/9606087 (1996).

[10] L.A. Kofman, A.D. Linde, and A.A. Starobinsky, Phys. Rev. Lett. **76**, 1011 (1996).

[11] E.T. Vishniac, K.A. Olive, and D.Seckel, Nucl. Phys. **B289**, 717 (1987).

[12] H.M. Hodges, and J.R. Primack, Phys. Rev. **D43**, 3155 (1991).

[13] M. Nagasawa and J. Yokoyama, Nucl. Phys. **B370**, 472 (1992).

[14] R. Basu, A.H. Guth, and A. Vilenkin, Phy. Rev. **D44**, 340 (1991).

[15] R.L. Davis and N. Turok, unpublished.

[16] B. Ratra, Phys. Rev **D31**, 1931 (1985).

[17] L. Dolan and R. Jackiw, Phys. Rev. **D9**, 3320 (1974).

[18] S. Coleman, R. Jackiw and H.D. Politzer, Phys. Rev. **D10**, 2491 (1975).

[19] S. Coleman and E. Weinberg, Phys. Rev. **D7**, 1888 (1973).

[20] L.F. Abbott, J.S. Kang and J. Schnitzer, Phys. Rev. **D13**, 2212 (1976).

[21] A.D. Linde, Nucl. Phys. **B125**, 369 (1977).

[22] N.D. Birrell and P.C.W. Davies, "*Quantum Fields in Curved Space*", Cambridge University Press, 1982.

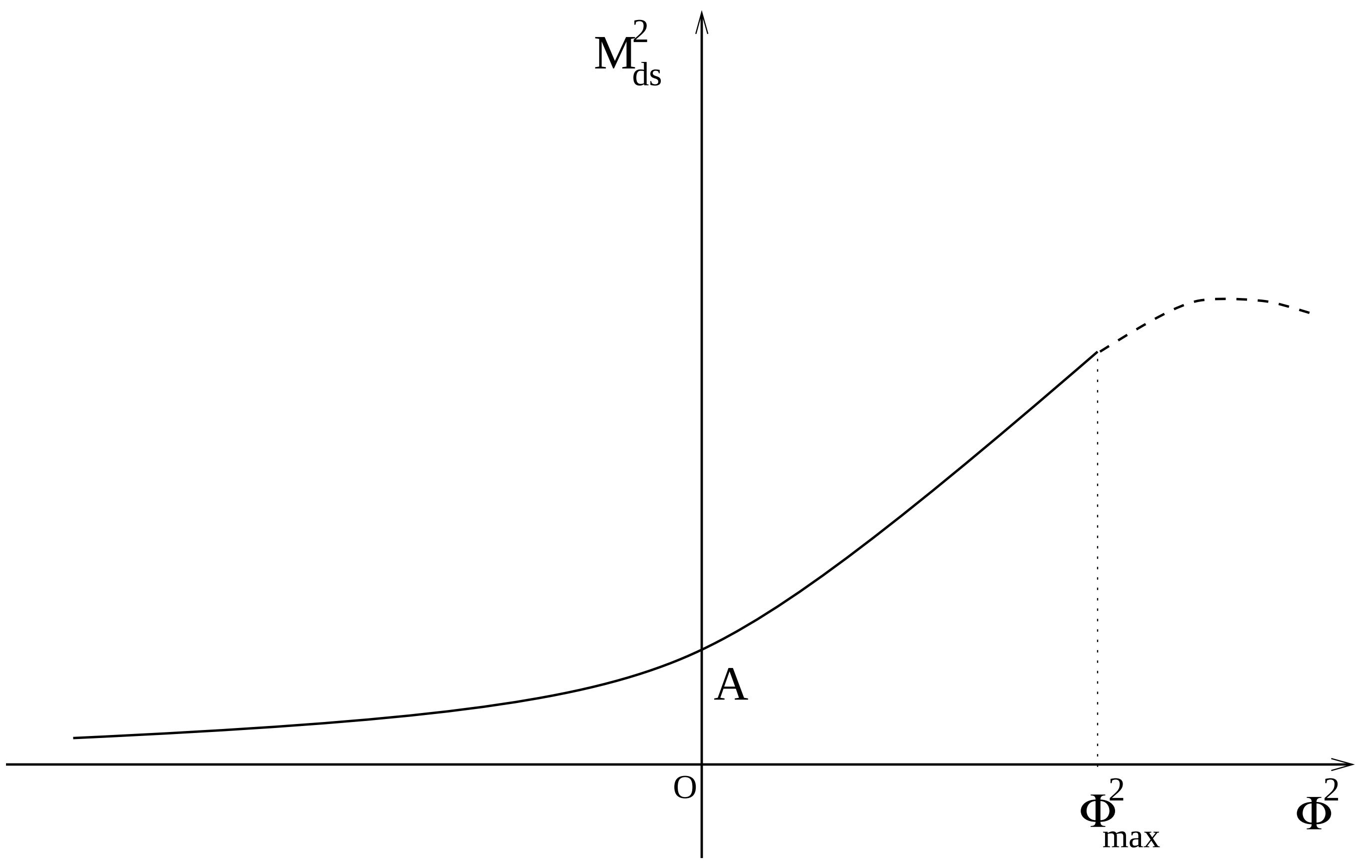
[23] T.S. Bunch and P.C.W. Davies, Proc. Roy. Soc. (London) **A360**, 117 (1978).

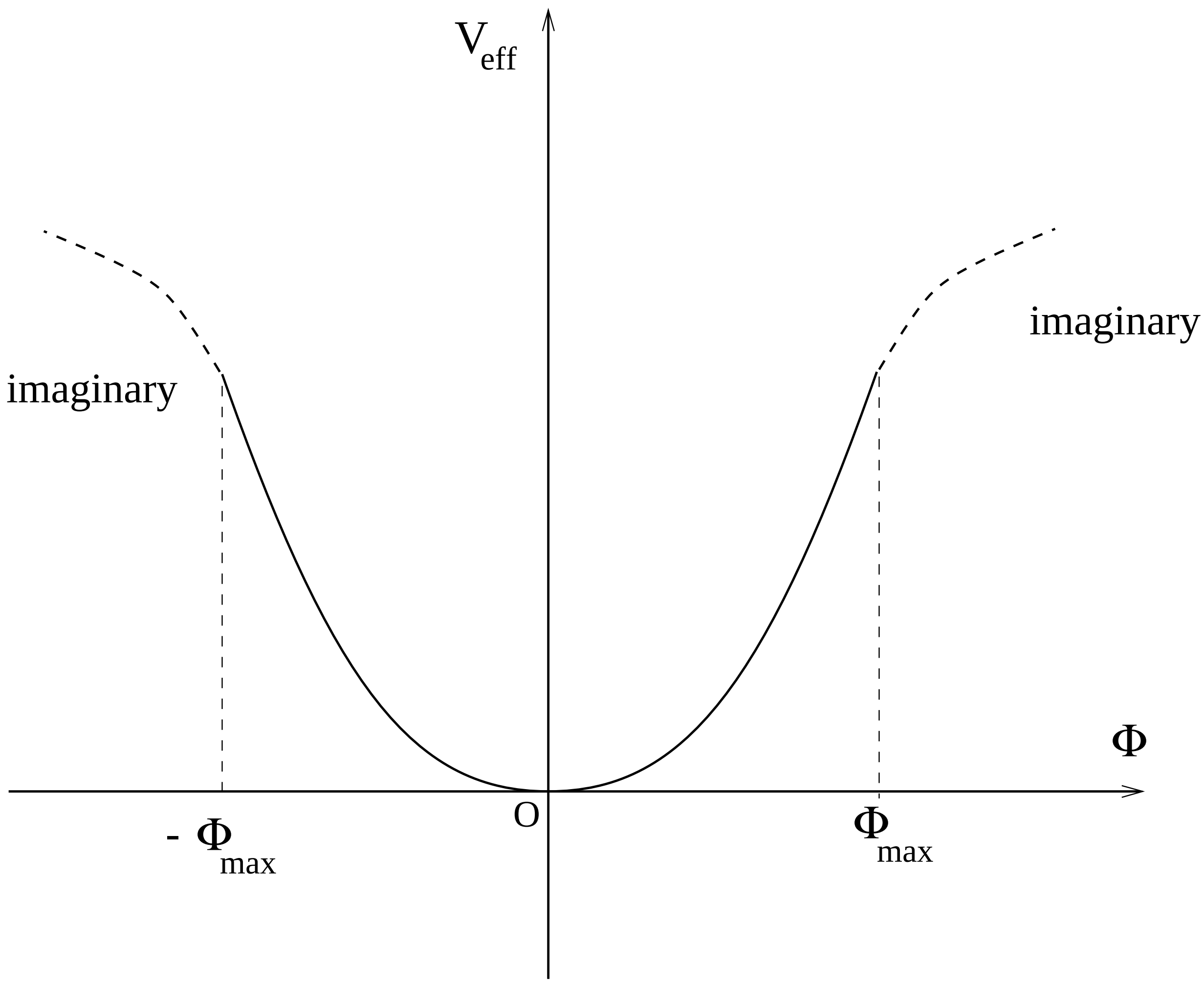
[24] P. Candelas and D.J. Raine, Phys. Rev. **D12**, 965 (1975).

[25] R. Jackiw, Phys. Rev. **D9**, 1686 (1974).

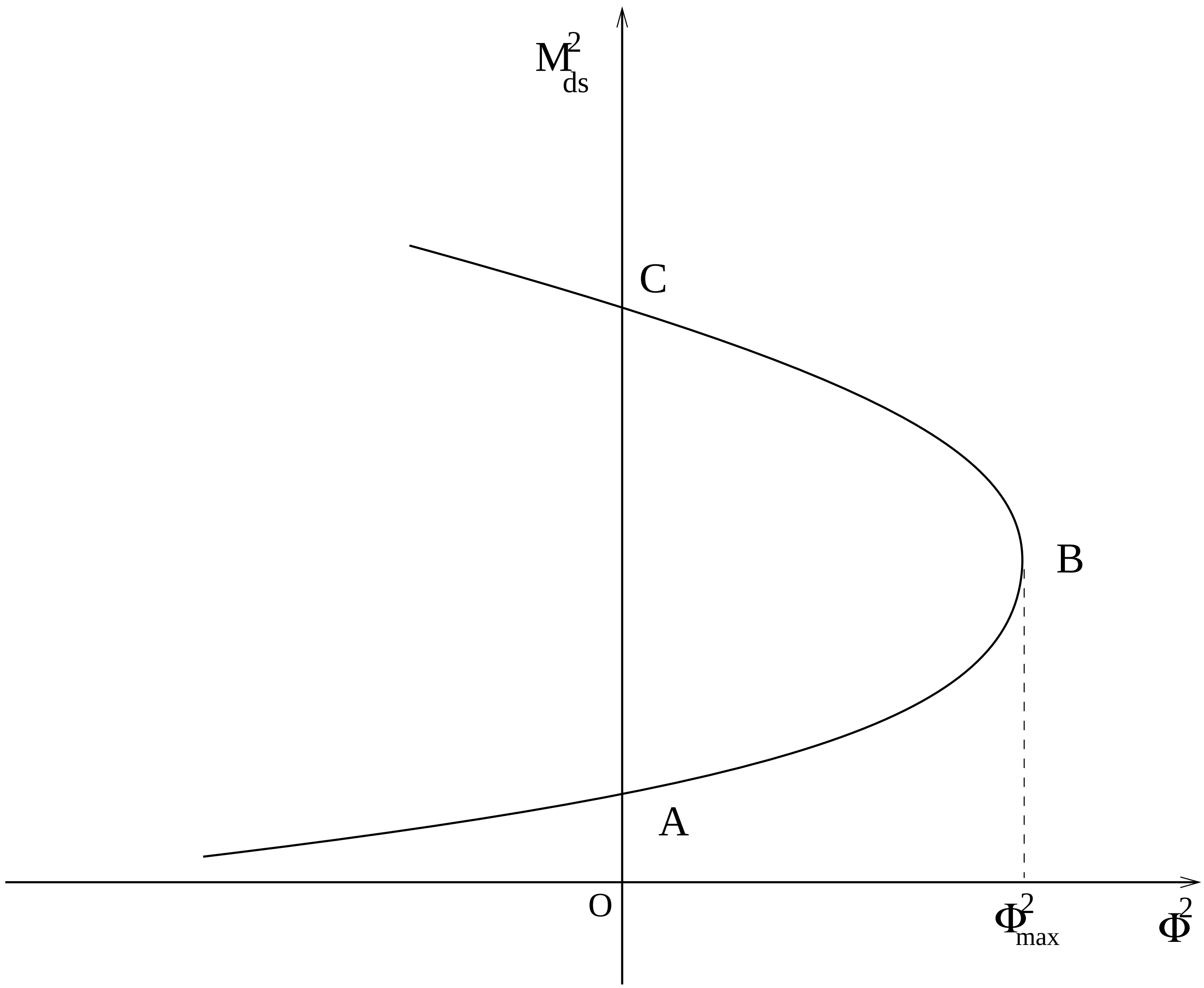
[26] A. Vilenkin, Nucl. Phys. **B226**, 504 (1983); *ibid*, 527 (1983).

[27] A.D. Linde, Phys. Lett. **114B**, 431 (1982).

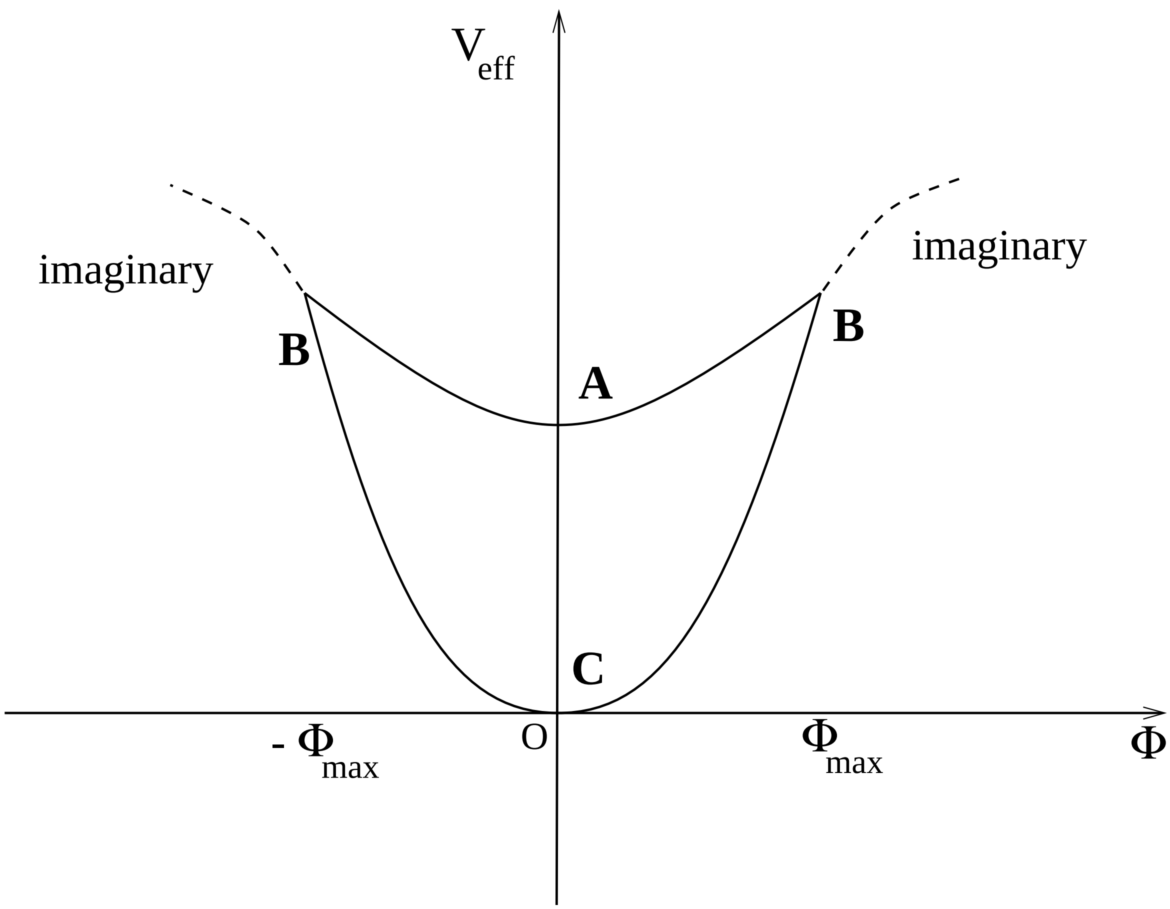

M_{ds}^2
A
O
Φ_{max}^2
Φ^2



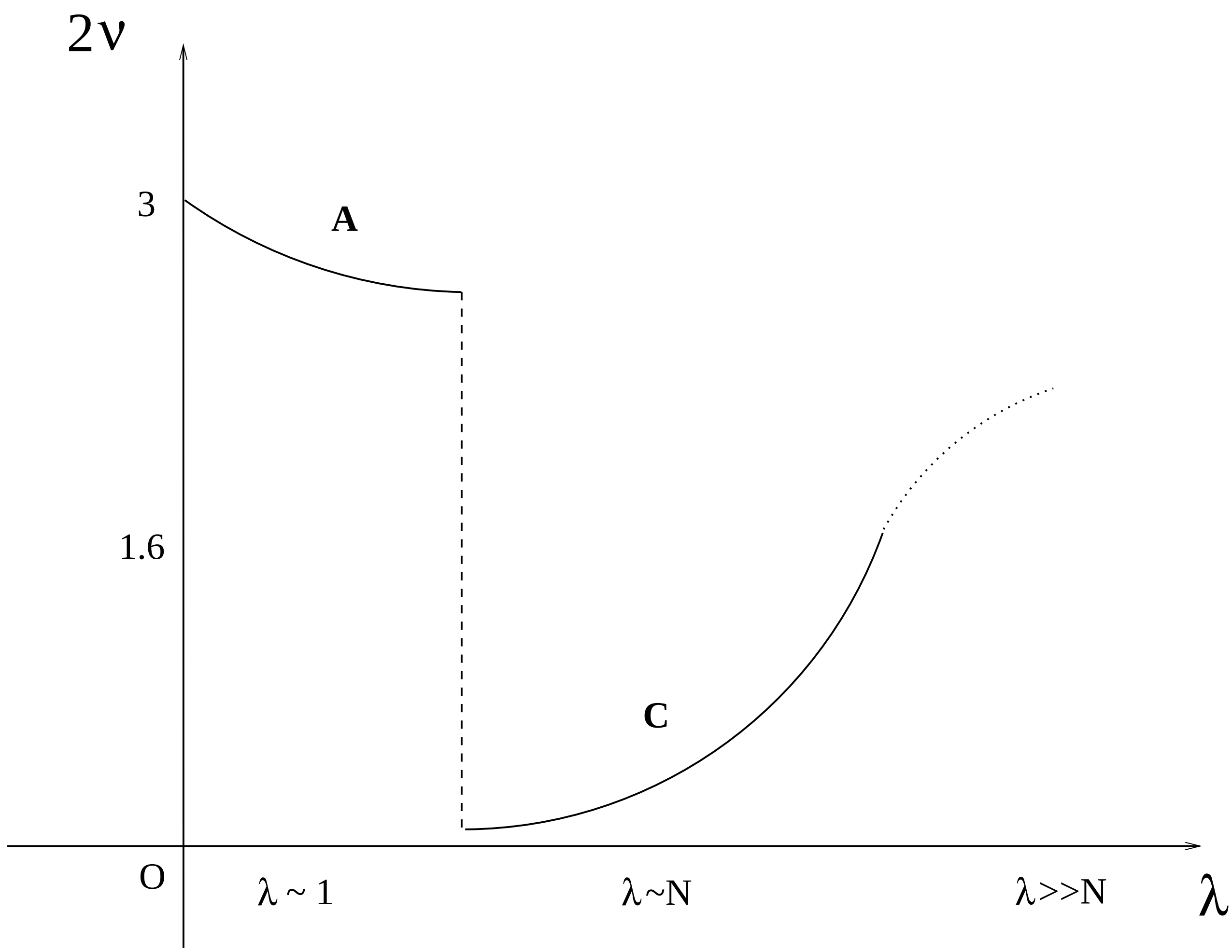
V_{eff}
imaginary
imaginary
Φ
$- \Phi_{max}$
O
Φ_{max}



M^2_{ds}
C
B
A
O
Φ^2_{max}
Φ^2



V_{eff}
imaginary
imaginary
B
B
A
C
- Φ_{max}
O
Φ_{max}
Φ



2ν
3
A
1.6
C
O
λ ~ 1
λ~N
λ>>N
λ